Continuum Modeling of the Dynamics of Externally Injection-Locked Coupled Oscillator Arrays

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Abstract—Mutually injection-locked arrays of electronic oscillators provide a novel means of controlling the aperture phase of a phased-array antenna, thus achieving the advantages of spatial power combining while retaining the ability to steer the radiated beam. In a number of design concepts, one or more of the oscillators are injection locked to a signal from an external master oscillator. The behavior of such a system has been analyzed by numerical solution of a system of nonlinear differential equations which, due to its complexity, yields limited insight into the relationship between the injection signals and the aperture phase. In this paper, we develop a continuum model, which results in a single partial differential equation for the aperture phase as a function of time. Solution of the equation is effected by means of the Laplace transformation and yields detailed information concerning the dynamics of the array under the influence of the external injection signals.

Index Terms— Beam steering, coupled oscillators, injection locked, phased array.

I. INTRODUCTION

A MUTUALLY coupled array of electronic oscillators can be made to oscillate in a mutually synchronized mode, in which the relative phases of the oscillators form a well-defined distribution useful in terms of exciting an array of radiating elements to achieve spatial power combining, beam formation, and steering of the radiated beam. Two methods of achieving this have been suggested. Having demonstrated that the ensemble of oscillators would oscillate at the average of the free-running (tuning) frequencies of the oscillators, Liao and York [1] showed that antisymmetrical detuning the oscillators at the ends of a linear array (or on the perimeter of a two-dimensional array) results in a linear phase progression across the array, which could be exploited to steer the beam without the use of phase shifters. Alternatively, Stephan [2] proposed and demonstrated externally injection locking the end oscillators with signals phase shifted with respect to each other to achieve a similar linear phase progression using only a single phase shifter. Such systems have, in the past, been analyzed by numerical solution of a system of nonlinear differential equations describing the oscillator coupling [2]–[4]. Pogorzelski *et al.* [5] have analyzed the
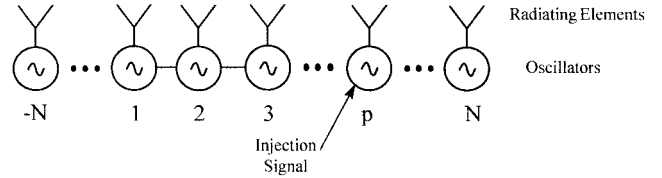
Fig. 1. A coupled oscillator array with the pth oscillator locked to an externally derived signal.

scheme of Liao and York using a continuum formulation involving a single partial differential equation for the aperture phase. That work shows that, in steady state, the behavior is governed by Poisson's equation of electrostatics in which the phase plays the role of the electrostatic potential and the detuning plays the role of electric charge density. In this paper, we develop a similar formalism for description of the behavior of such arrays in the presence of externally derived injection signals [6] and, thus, provide analytical tools appropriate to the description of systems such as that proposed by Stephan [2].

The array to be studied is illustrated schematically in Fig. 1, a one-dimensional array wherein the pth oscillator is externally injection locked. As mentioned above, the behavior of such arrays of oscillators has been described in detail using a coupled set of nonlinear differential equations [3], [4]. These equations are derived by first describing the behavior of an individual oscillator with injection locking in the manner of Adler [7] and then allowing the injection signals to be provided by the neighboring oscillators in the array. This formulation, when applied to the array shown in Fig. 1, results in the coupled set

$$\frac{d\theta_i}{dt} = \omega_{\text{tune},\,i} - \sum_{j=-N}^{N} \Delta\omega_{\text{lock},\,ij} \sin(\Phi_{ij} + \theta_i - \theta_j)$$
$$- \delta_{ip}\Delta\omega_{\text{lock},\,p,\,\text{inj}} \sin(\theta_p - \theta_{\text{inj}}) \quad (1)$$

for $i = -N, -N+1, \cdots, 0, 1, 2, \cdots N$. Φ_{ij} is the phase of the injection signal from oscillator j evaluated at oscillator i, the coupling phase, and $\varepsilon_{ij}d_j$ is the amplitude of this signal while the interoscillator locking range is defined by

$$\Delta\omega_{\text{lock},\,ij} = \frac{\varepsilon_{ij}\omega_{\text{tune},\,i}}{2Q}\frac{\alpha_j}{\alpha_i} \quad (2)$$

where Q is the quality factor of the oscillators and $\omega_{\text{tune},\,i}$ is the free-running frequency of the ith oscillator. α_i is the amplitude of the output signal of the ith oscillator. A corresponding locking range $\Delta\omega_{\text{lock},\,p,\,\text{inj}}$ is similarly defined for the external injection signal. The phase θ_i is the phase of

the ith oscillator, i.e.,

$$\theta_i = \omega_{\mathrm{ref}}t + \phi_i \tag{3}$$

where ω_{ref} is the reference frequency for defining the phase of each oscillator. Applying this to a one-dimensional array and following [3], we assume only nearest neighbor coupling, zero coupling phase, and that all of the interoscillator locking ranges are identical. This leads to

$$\frac{d\theta_i}{dt} = \omega_{\mathrm{tune},\,i} - \Delta\omega_{\mathrm{lock}} \sum_{\substack{j=i-1\\j\neq i}}^{j=i+1} \sin(\theta_i - \theta_j) - \delta_{ip}\Delta\omega_{\mathrm{lock},\,p,\,\mathrm{inj}}$$
$$\cdot \sin(\theta_p - \theta_{\mathrm{inj}}) \tag{4}$$

which describes a one-dimensional array with one externally derived injection signal. If additional signals are injected at other oscillators, one merely appends additional terms to the right-hand side, with corresponding Kroneker delta functions denoting the locations of the corresponding oscillators.

In this paper, we focus on development of a corresponding continuum formulation in which the phase is described by a continuous function passing through, at each oscillator, a value equal to the phase of that oscillator and governed by a single partial differential equation. By solving this equation, we exhibit the dynamic behavior of linear coupled oscillator arrays under external injection locking.

II. DERIVATION OF THE CONTINUUM MODEL

Following [5], assuming that the interoscillator phase differences are small, we approximate the sine function by its argument, thus obtaining

$$\frac{d\theta_i}{dt} = \omega_{\mathrm{tune},\,i} - \Delta\omega_{\mathrm{lock}} \sum_{\substack{j=i-1\\j\neq i}}^{j=i+1} (\theta_i - \theta_j)$$
$$= \omega_{\mathrm{tune},\,i} - \Delta\omega_{\mathrm{lock}}(\theta_i - \theta_{i+1} + \theta_i - \theta_{i-1})$$
$$- \delta_{ip}\Delta\omega_{\mathrm{lock},\,p,\,\mathrm{inj}}(\theta_p - \theta_{\mathrm{inj}}) \tag{5}$$

which, using (3), can be rewritten in the form

$$\frac{d\phi_i}{dt} = \omega_{\mathrm{tune},\,i} - \omega_{\mathrm{ref}} + \Delta\omega_{\mathrm{lock}}(\phi_{i+1} - 2\phi_i + \phi_{i-1})$$
$$- \delta_{ip}\Delta\omega_{\mathrm{lock},\,p,\,\mathrm{inj}}(\phi_p - \phi_{\mathrm{inj}}) \tag{6}$$

for $i = -N, -N+1, \cdots, 0, 1, 2, \cdots, N$. At this point, we note that the quantity in the first pair of parentheses is merely a finite-difference approximation for the second derivative of the phase with respect to a spatial variable x, which corresponds to the index i at integer values. Thus, (6) can now be easily recognized as the finite-difference approximation corresponding to the partial differential equation

$$\frac{\partial^2\phi}{\partial x^2} - \frac{\partial\phi}{\partial\tau} = -\frac{\omega_{\mathrm{tune}} - \omega_{\mathrm{ref}}}{\Delta\omega_{\mathrm{lock}}} + \delta_{ip}\frac{\Delta\omega_{\mathrm{lock},\,p,\,\mathrm{inj}}}{\Delta\omega_{\mathrm{lock}}}(\phi - \phi_{\mathrm{inj}}) \tag{7}$$

for $-a-1/2 \le x \le a+1/2$ where $\phi(x, \tau)$ is the phase across the array and the unitless time τ is the time t multiplied by the locking range $\Delta\omega_{\mathrm{lock}}$. The points $x = \pm a$ correspond to the index values $i = \pm N$. The array extends over $2N+1$ unit

cells with an oscillator at the center of each unit cell leading to the range of x noted above.

Now, let the injection signal be represented by $V(x)\phi_{\mathrm{inj}}(\tau)$, where the time dependence of the injection signal phase is given by ϕ_{inj} and the spatial distribution of the injection signals is given by V. In the case of (7),

$$V(x) = \frac{\Delta\omega_{\mathrm{lock},\,p,\,\mathrm{inj}}}{\Delta\omega_{\mathrm{lock}}}\delta(x - p) \tag{8}$$

where we have replaced the Kronecker delta with the Dirac delta. Here, again, for analytical convenience, as was done in [5], the Dirac delta is used instead of a pulse one unit cell wide. Equation (7) then becomes

$$\frac{\partial^2\phi}{\partial x^2} - V(x)\phi - \frac{\partial\phi}{\partial\tau} = -\frac{\omega_{\mathrm{tune}} - \omega_{\mathrm{ref}}}{\Delta\omega_{\mathrm{lock}}} - V(x)\phi_{\mathrm{inj}}(\tau). \tag{9}$$

Note that the driving function is the distribution of the oscillator free-running (tuning) frequencies relative to the reference frequency, plus the injection signals. This equation forms the basis of the remainder of the analysis presented here.

A. The Infinite-Length Array

Consider now an array for which $a = \infty$, i.e., a linear array of infinite length. Let all of the oscillators be tuned to the same frequency ω_0. Now, inject an externally derived signal of frequency $\omega_0 + C_0\Delta\omega_{\mathrm{lock}}u(\tau)$, where u is the unit step function, into the oscillator at $x = b$. Thus, prior to $t = 0$, all the oscillators will be in phase. Subsequent to $t = 0$, the behavior of the oscillators will be given by the solution to the partial differential equation

$$\frac{\partial^2\phi}{\partial x^2} - C\delta(x - b)\phi - \frac{\partial\phi}{\partial\tau} = -C_0 C\delta(x - b)\tau u(\tau) \tag{10}$$

where ω_{ref} has been chosen to be ω_0. We begin by Laplace transformation with respect to τ. The transformed equation takes the form

$$\frac{\partial^2 F}{\partial x^2} - C\delta(x - b)F - sF = -\frac{C_0}{s^2}C\delta(x - b). \tag{11}$$

Now, by defining,

$$\tilde{F} = F - \frac{C_0}{s^2} \tag{12}$$

one obtains

$$\frac{\partial^2\tilde{F}}{\partial x^2} - C\delta(x - b)\tilde{F} - s\tilde{F} = \frac{C_0}{s}. \tag{13}$$

A particular integral of this equation is

$$\tilde{F}_p = -\frac{C_0}{s^2} \tag{14}$$

for $x \neq b$. We postulate a solution of the homogeneous equation of the form

$$\tilde{F}_h = C_1 e^{-\sqrt{s}|x-b|}. \tag{15}$$

The unknown constant C_1 can be determined by imposing on the sum of the homogeneous and particular solutions the

appropriate slope discontinuity at $x = b$, i.e.,

$$\frac{d\tilde{F}}{dx}\Bigg|_{x=b^-}^{x=b^+} = \tilde{F}(b) \tag{16}$$

which implies that

$$C_1 = \frac{C_0 C}{s^2(2\sqrt{s} + C)} \tag{17}$$

so that the solution of (11) with (12) becomes

$$\tilde{F}(x) = \frac{C_0}{s^2}\left[\frac{C}{2\sqrt{s} + C}e^{-\sqrt{s}|x-b|} - 1\right]. \tag{18}$$

This is the Laplace transform of the phase distribution over the infinite array. The corresponding frequency distribution is given by

$$\tilde{W}(x) = \frac{C_0}{s}\left[\frac{C}{2\sqrt{s} + C}e^{-\sqrt{s}|x-b|} - 1\right]. \tag{19}$$

Defining

$$\tilde{\omega}(x, \tau) = \omega(x, \tau) - C_0 u(\tau) = \frac{\partial\phi}{\partial\tau} - C_0 u(\tau) \tag{20}$$

the inverse Laplace transform of (19) is

$$\tilde{\omega}(x, \tau) = C_0\left[\text{erfc}\left(\frac{|x-b|}{2\sqrt{\tau}}\right) - e^{C|x-b|/2}e^{C^2\tau/4}\right.$$
$$\left.\cdot\,\text{erfc}\left(C\frac{\sqrt{\tau}}{2} + \frac{|x-b|}{2\sqrt{\tau}}\right) - 1\right]. \tag{21}$$

For each fixed value of x, this function begins at a value of $-C_0$ when $\tau = 0$ and evolves smoothly and monotonically toward a final value of zero at infinite τ. This transition from the reference frequency to the injection frequency occurs first for x values near zero and later for oscillators more distant from the center of the array. This is, of course, to be expected since the effect of the sudden switch in injection signal frequency would be expected to diffuse from the injection point outward in both directions along the array. The diffusion rate is governed by the ratio of τ to t, i.e., the interoscillator locking range. The phase behavior of the array can be obtained as the time integral of this function. Interestingly, however, the resulting function approaches infinity for infinite τ, indicating that the phase never reaches a steady-state value, as does the frequency. Rather, it continues to evolve for all time. Specifically, for late times, the frequency differs from the injection frequency as one over the square root of the time, which implies that the phase, which is its time integral, differs from the injection phase as the square root of time which, of course, approaches infinity for infinite time.

Fig. 2 shows the result of a numerical evaluation of (21) with $C_0 = 1$. This graph indicates that the center oscillator approaches the injection frequency most rapidly and the others
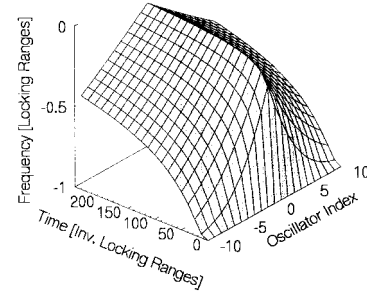

Fig. 2. Frequencies of the oscillators in an infinite linear array under external injection locking of one oscillator (at $x = 0$) with a step frequency change.
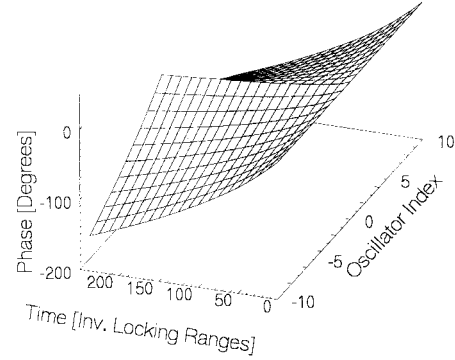

Fig. 3. Phases of the oscillators of Fig. 2.

follow at later times as would be expected. Integrating this frequency function with respect to time yields the phase function, shown in Fig. 3, where the definition of $\tilde{\omega}$ suppresses the linear time dependence arising from the frequency transition.

B. The Finite-Length Array

Consider now an array extending from $-a$ to a in x, thus having $2a + 1$ oscillators. To derive the dynamic behavior of the phase in such an array with the element at $x = b$ externally injection locked, we must effectively add homogeneous solutions of (11) and (13) to the particular integral (18) so as to simultaneously satisfy both the boundary conditions at the ends of the array and the slope discontinuity condition at $x = b$. It was shown in [5] that the boundary conditions at the array ends $x = a + 1/2$ and $x = -a - 1/2$ are the classical Neumann conditions independent of time. Following the prescription suggested above, we postulate a solution of the form

$$\tilde{F}(x, s) = C_b e^{-|x-b|\sqrt{s}} + C_R e^{-x\sqrt{s}} + C_L e^{x\sqrt{s}} - \frac{C_0}{s^2}. \tag{22}$$

One can then determine the unknown constants C_b, C_R, and C_L by imposing Neumann boundary conditions at the array ends together with the appropriate slope discontinuity at $x = b$. (Note that C_b will not be the same as C_1 derived for the infinite

$$\tilde{F}(x, s) = \frac{C_0}{s^2}\left\{\frac{C\cosh\left[\sqrt{s}(2a + 1 - |b - x|)\right] + C\cosh\left[\sqrt{s}(b + x)\right]}{2\sqrt{s}\sinh\left[\sqrt{s}(2a + 1)\right] + C\cosh\left[\sqrt{s}(2b)\right] + C\cosh\left[\sqrt{s}(2a + 1)\right]}\right\} - \frac{C_0}{s^2} \tag{23}$$

array.) The resulting solution, using (12), is shown in (23), at the bottom of the previous page.

Despite the presence of $\sqrt{s}$, there is no branch cut in the s plane because $\tilde{F}(x, s)$ is an even function of $\sqrt{s}$. Thus, the inverse Laplace transform can be obtained as the sum of the residues at the poles, all of which are located on the negative real axis. Of course, some care must be taken concerning the double pole at the origin, which gives rise to a term linear in time. This term arises because of the transition in the ensemble frequency as it follows the injection signal frequency, resulting in a steady-state phase variation, which is linear in time. Aside from the double pole at the origin, the pole closest to the origin determines the dominant transient response time constant for the array. The location of this pole can be estimated by approximate solution of the transcendental equation obtained by setting the denominator of (23) equal to zero. The result is

$$\sigma_{\min} \approx \left(\frac{\pi}{2(a+|b|)+1} \right)^2. \tag{24}$$

That is, the response time is roughly proportional to the *square* of the number of elements in the array between the injection point and the farthest end.

The frequency is again obtained by multiplication by s, i.e., as shown in (25), at the bottom of this page. As in the infinite array case, the inverse transform of this function approaches zero at infinite time. However, unlike the infinite case, the approach is exponential instead of $1/\sqrt{\tau}$. Correspondingly, aside from the linear time dependence, the phase approaches a temporal constant at infinite time, but this temporal constant depends on x parabolically. This can be seen by invoking the final value theorem and noting that

$$\lim_{s \to 0}\{s\tilde{F}(x, s)\}$$
$$= \lim_{\tau \to \infty} \{\phi(x, \tau) - C_0\tau\}$$
$$= \frac{C_0}{2}\left[(x^2 - b^2) - (2a+1)|b-x| - \frac{2}{C}(2a+1)\right]. \tag{26}$$

Note that the discontinuity in phase slope across the injection point is $(2a+1)C_0$. By integrating (9) across the injection point, one finds that this discontinuity must be equal to C times the phase difference between the injection signal and the output of the injected oscillator, i.e., the one at $x = b$. This phase difference must be small for the present linearized theory to apply. In fact, recalling that this phase difference replaces the sine of this phase difference in the original nonlinear theory, we find that $(2a+1)C_0$ must be less than C to maintain phase lock. Therefore, we arrive at the requirement that

$$C_0 < \frac{C}{2a+1} \tag{27}$$

to maintain phase lock in steady state. That is, as can been noted using the nonlinear discrete model of York [3], the
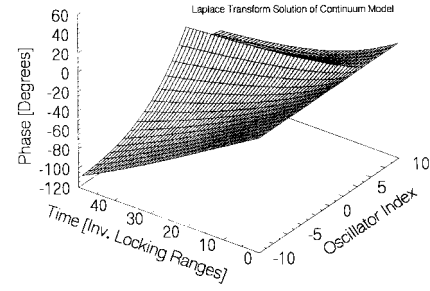

Fig. 4. Phases of the oscillators in a linear array of 21 oscillators in which oscillator 5 is injection locked to an externally derived signal stepped in frequency at time zero.
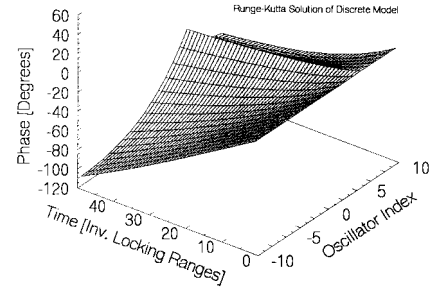

Fig. 5. Oscillator phases for the array of Fig. 4 computed via the discrete model.
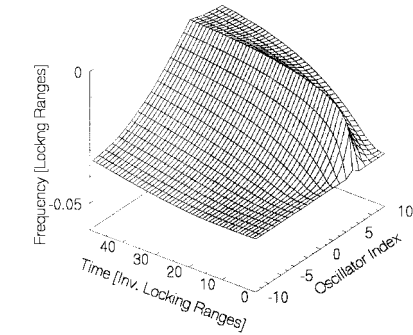

Fig. 6. Frequencies of the oscillators of Fig. 8.

frequency shift, which can be induced by a step change in the frequency injected at one oscillator of the array is limited, not just to the locking range, but to the locking range divided by the number of oscillators in the array. Fig. 4 shows the calculated phase variation, suppressing the linear dependence, for an example in which $a = 10$, $b = 5$, and C is unity. According to (27), C_0 must be less than 1/21. Here, we choose it to be 0.04. As a validation, the same case was computed via Runge–Kutta solution [8] of the nonlinear discrete model equations and the result, shown in Fig. 5, is indistinguishable from that of the continuum formulation. Fig. 6 shows the corresponding frequency variation.

$$\tilde{W}(x, s) = \frac{C_0}{s}\left\{ \frac{C\cosh\left[\sqrt{s}(2a+1-|b-x|)\right] + C\cosh\left[\sqrt{s}(b+x)\right]}{2\sqrt{s}\sinh\left[\sqrt{s}(2a+1)\right] + C\cosh\left[\sqrt{s}(2b)\right] + C\cosh\left[\sqrt{s}(2a+1)\right]} \right\} - \frac{C_0}{s} \tag{25}$$

If the injection point is located at the center of the array, i.e., if $b = 0$, then (23) and (25) simplify to

$$
\begin{aligned}
\tilde{F}(x, s) &= \frac{C_0}{s^2} \left\{ \frac{C \cosh\left[\sqrt{s}\left(a+\frac{1}{2}-|x|\right)\right]}{2\sqrt{s} \sinh\left[\sqrt{s}\left(a+\frac{1}{2}\right)\right] + C \cosh\left[\sqrt{s}\left(a+\frac{1}{2}\right)\right]} \right\} \\
&\quad - \frac{C_0}{s^2}
\end{aligned}
\tag{28}
$$

and

$$
\begin{aligned}
\tilde{W}(x, s) &= \frac{C_0}{s} \left\{ \frac{C \cosh\left[\sqrt{s}\left(a+\frac{1}{2}-|x|\right)\right]}{2\sqrt{s} \sinh\left[\sqrt{s}\left(a+\frac{1}{2}\right)\right] + C \cosh\left[\sqrt{s}\left(a+\frac{1}{2}\right)\right]} \right\} \\
&\quad - \frac{C_0}{s}.
\end{aligned}
\tag{29}
$$

The behavior obtained if all of the oscillators are externally injection locked to the same signal can be determined by returning to (9) and setting $V = C$. This results in

$$
\frac{\partial^2 \phi}{\partial x^2} - C\phi - \frac{\partial \phi}{\partial \tau} = -\frac{\omega_{\text{tune}} - \omega_{\text{ref}}}{\Delta\omega_{\text{lock}}} - C\phi_{\text{inj}}(\tau).
\tag{30}
$$

Laplace transformation leads to

$$
\frac{\partial^2 \tilde{F}}{\partial x^2} - (C+s)\tilde{F} = \frac{C_0}{s}
\tag{31}
$$

where $\tilde{F}$ is again given by (12). This equation has a solution of the form

$$
\tilde{F}(x, s) = -\frac{C_0}{s(C+s)} + C_R e^{-x\sqrt{C+s}} + C_L e^{x\sqrt{C+s}}.
\tag{32}
$$

Applying the Neumann boundary conditions at the ends of the array, we find that C_R and C_L are both zero so that

$$
F(x, s) = \frac{C_0}{s^2} - \frac{C_0}{s(C+s)} = \frac{C_0}{s^2} - \frac{C_0}{Cs} - \frac{C_0}{C(C+s)}
\tag{33}
$$

and

$$
\phi(x, \tau) = C_0 \tau u(\tau) - \frac{C_0}{C}(1 - e^{-C\tau})u(\tau).
\tag{34}
$$

Upon differentiation with respect to time, we find that

$$
\omega(x, \tau) = \omega_0 + C_0(1 - e^{-C\tau})u(\tau)
\tag{35}
$$

which indicates that the frequency of all the oscillators simultaneously evolves from the initial value ω_0 to the final value $\omega_0 + C_0$ with a time constant

$$
\frac{C}{\Delta\omega_{\text{lock}}} = \frac{1}{\Delta\omega_{\text{inj}}}
\tag{36}
$$

where $\Delta\omega_{\text{inj}}$ is the locking range of the externally injected oscillator in the array. That is, the time constant is propor-

tional to the injection signal locking range rather than the interoscillator locking range. Moreover, unlike in (24), there is no dependence on the number of oscillators in the array.

Finally, we emphasize that the continuous function $\phi(x, \tau)$ only has meaning at integer values of x where it takes on the value of the phase of the oscillator of index $n = x$. The above development was carried out for an odd number of oscillators. If the number of oscillators is even, the theory as developed can be applied by setting $a = M/2$ where M is the number of oscillators. In that case, however, the solution $\phi(x, \tau)$ only has meaning when x is a half-integer where it takes on the value of the phase of the oscillator indexed by that value of x.

III. THE DYNAMICS OF BEAM STEERING

A. Step Phase Shift

Suppose now that two of the oscillators in the array are injection locked to externally derived signals of the same frequency, but differing phase. Such an arrangement was proposed by Stephan as a means of steering the radiated beam [2]. Equation (9) then becomes

$$
\begin{aligned}
\frac{\partial^2 \phi}{\partial x^2} &- \left[B_1\delta(x - b_1) + B_2\delta(x - b_2)\right]\phi - \frac{\partial \phi}{\partial \tau} \\
&= -B_1\delta(x - b_1)p_1 u(\tau) - B_2\delta(x - b_1)p_2 u(\tau)
\end{aligned}
\tag{37}
$$

where the B's measure the strengths of the two injection signals, the b's are their locations, and the p's are their phases. Laplace transformation yields

$$
\begin{aligned}
\frac{\partial^2 F}{\partial x^2} &- \left[B_1\delta(x - b_1) + B_2\delta(x - b_2)\right]F - sF \\
&= -B_1\delta(x - b_1)\frac{p_1}{s} - B_2\delta(x - b_1)\frac{p_2}{s}.
\end{aligned}
\tag{38}
$$

A solution of this equation is now postulated in the form

$$
\begin{aligned}
F(x, s) = C_1 e^{-|x-b_1|\sqrt{s}} &+ C_2 e^{-|x-b_2|\sqrt{s}} + C_R e^{-x\sqrt{s}} \\
&+ C_L e^{x\sqrt{s}}
\end{aligned}
\tag{39}
$$

and the unknown constants are determined by application of the Neumann boundary conditions at the ends of the array and the slope discontinuities at the two injection points. That is, letting $2h = 2a + 1$, we have

$$
\begin{aligned}
&-\sqrt{s}C_1 e^{-\sqrt{s}(h-b_1)} - \sqrt{s}C_2 e^{-\sqrt{s}(h-b_2)} - C_R e^{-\sqrt{s}h} + C_L e^{\sqrt{s}h} \\
&\quad = 0 \\
&\sqrt{s}C_1 e^{-\sqrt{s}(h+b_1)} + \sqrt{s}C_2 e^{-\sqrt{s}(h+b_2)} - C_R e^{\sqrt{s}h} + C_L e^{-\sqrt{s}h} \\
&\quad = 0 \\
&-2\sqrt{s}C_1 - B_1\left[C_1 + C_2 e^{-\sqrt{s}(b_2-b_1)} + C_R e^{-\sqrt{s}b_1} + C_L e^{\sqrt{s}b_1}\right] \\
&\quad = -\frac{B_1 p_1}{s} \\
&-2\sqrt{s}C_2 - B_2\left[C_1 e^{-\sqrt{s}(b_2-b_1)} + C_2 + C_R e^{-\sqrt{s}b_2} + C_L e^{\sqrt{s}b_2}\right] \\
&\quad = -\frac{B_2 p_2}{s}.
\end{aligned}
\tag{40}
$$

Solving these equations simultaneously and substituting into (39) yields the solution

$$
\begin{aligned}
F(x, s) &= \frac{1}{s\Delta}\Bigg\{ 2B_2 p_2 \cosh\left[\sqrt{s}(2h-|b_2-x|)\right] \\
&\quad + 2B_2 p_2 \cosh\left[\sqrt{s}(b_2+x)\right] \\
&\quad + 2B_1 p_1 \cosh\left[\sqrt{s}(2h-|b_1-x|)\right] \\
&\quad + 2B_1 p_1 \cosh\left[\sqrt{s}(b_1+x)\right] \\
&\quad + \frac{B_1 B_2 p_2}{\sqrt{s}} \sinh\left[\sqrt{s}(2h-|b_2-x|)\right] \\
&\quad - \frac{B_1 B_2 p_1}{\sqrt{s}} \sinh\left[\sqrt{s}(2h-(b_2-b_1)-|b_2-x|)\right] \\
&\quad - \frac{B_1 B_2 p_2}{\sqrt{s}} \sinh\left[\sqrt{s}(2b_1-|b_2-x|)\right] \\
&\quad - \frac{B_1 B_2 p_1}{\sqrt{s}} \sinh\left[\sqrt{s}((b_2+b_1)-|b_2-x|)\right] \\
&\quad + \frac{B_1 B_2 p_1}{\sqrt{s}} \sinh\left[\sqrt{s}(2h-|b_1-x|)\right] \\
&\quad - \frac{B_1 B_2 p_2}{\sqrt{s}} \sinh\left[\sqrt{s}(2h-(b_2-b_1)-|b_1-x|)\right] \\
&\quad + \frac{B_1 B_2 p_1}{\sqrt{s}} \sinh\left[\sqrt{s}(2b_2-|b_1-x|)\right] \\
&\quad + \frac{B_1 B_2 p_2}{\sqrt{s}} \sinh\left[\sqrt{s}((b_2+b_1)+|b_1-x|)\right] \Bigg\} \quad (41)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta &= 4\sqrt{s}\, \sinh\left[\sqrt{s}(2h)\right] + 2B_2 \cosh\left[\sqrt{s}(2b_2)\right] \\
&\quad + 2B_1 \cosh\left[\sqrt{s}(2b_1)\right] + 2(B_2+B_1)\cosh\left[\sqrt{s}(2h)\right] \\
&\quad + \frac{B_1 B_2}{\sqrt{s}}\Big\{ \sinh\left[\sqrt{s}(2h)\right] - \sinh\left[\sqrt{s}(2b_1)\right] + \sinh\left[\sqrt{s}(2b_2)\right] \\
&\quad - \sinh\left[\sqrt{s}(2h-2(b_2-b_1))\right]\Big\}.
\end{aligned} \quad (42)
$$

The inverse transform can again be found using residue calculus. Here, again, the poles all lie on the negative real axis.

Using the final value theorem, we find that the steady-state phase distribution is as shown in (43), at the bottom of this page, which, of course, is just the residue at the pole at $s=0$. Note that there is no constraint corresponding to (27) here because the injection signals have the same frequency as the array; they are merely shifted in phase. The phase shift must only be confined to less than 90° to maintain lock. This would appear, at first glance, to be a serious drawback associated with this beam-steering technique. However, as will be seen, this
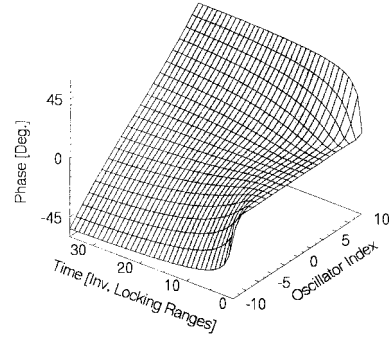

Fig. 7. Phases of the oscillators in a linear array of 21 oscillators in which the end oscillators are injection locked to externally derived signals out of phase by ±60°.

limit only applies if the phase is changed stepwise in time. If a gradual phase shift is introduced, the final value is theoretically limited only to 90° times the number of oscillators in the array less one.

As an example of beam steering, we choose a case where a 21-oscillator array is injection locked at the ends with signals having equal amplitude and antisymmetric phase. That is,

$$
\begin{aligned}
B_1 &= B_2 = 1 \\
b_1 &= -h \\
b_2 &= h \\
p_1 &= -60° \\
p_2 &= 60°.
\end{aligned} \quad (44)
$$

The resulting dynamic behavior of the oscillator phases is shown in Fig. 7. When this phase distribution is applied to a 21-element linear array of radiating elements separated by a half-wavelength, the resulting steering angle is only about 0.6 beamwidths. Clearly, greater phase shift is needed. This will be addressed below.

B. Gradual Phase Shift

If wide angle scanning is desired, large phase shifts must be produced. This requires a gradual shift of the phase of the injected signals if lock is to be maintained. Such a case was presented by Stephan [2] and the present theory can be used to reproduce his results as follows. The solution represented by (41) and (42) corresponds to a step change in phase of the injection signals. By convolving this step function with a Gaussian, the transition can be made gradual. The corresponding solution for the phase can be obtained by convolving the step solution with the same Gaussian. Since, in the time domain, the solution is expressed as a sum of exponentials, one need only convolve each exponential with the Gaussian. This convolution can be written as multiplication by an expression in terms of complementary error functions.

$$
\phi(x, \infty) = \frac{B_2 p_2 + B_1 p_1 + \frac{1}{2}B_1 B_2\left[(b_2-b_1)(p_2+p_1) + (|b_1-x|-|b_2-x|)(p_2-p_1)\right]}{B_2 + B_1 + B_1 B_2(b_2-b_1)} \quad (43)
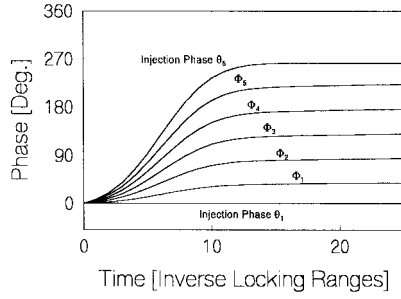$$

Fig. 8. Phases of the oscillators in a linear array of five oscillators, in which the end oscillators are injection locked to externally derived signals gradually shifted out of phase to a maximum of 270°.
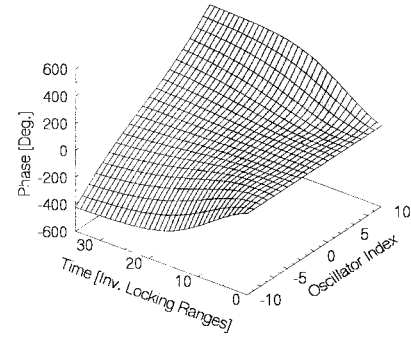

Fig. 9. Phases of the oscillators in a linear array of 21 oscillators in which the end oscillators are injection locked to externally derived signals gradually shifted out of phase to a maximum of 1200°.

Let the Gaussian be written in the form

$$g(\tau) = e^{-\alpha(\tau - \tau_0)^2}. \tag{45}$$

The convolution of this function with an exponential can then be written in the form

$$A_n e^{-\sigma_n \tau} * g(\tau)$$
$$= \int_0^\tau A_n e^{-\sigma_n(\tau - \mu)} e^{-\alpha(\mu - \tau_0)^2} d\mu$$
$$= A_n e^{-\sigma_n \tau} \left[e^{-\sigma_n \tau_0} e^{\sigma_n^2/(4\alpha)} \frac{1}{\sqrt{\alpha}} \int_{v_1}^{v_2} e^{-v^2} dv \right] \tag{46}$$

where

$$v_1 = -\sqrt{\alpha}\left(\tau_0 + \frac{\sigma_n}{2\alpha}\right)$$
$$v_2 = \sqrt{\alpha}\left[\tau - \left(\tau_0 + \frac{\sigma_n}{2\alpha}\right)\right]. \tag{47}$$

In terms of the complementary error function defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \tag{48}$$

this expression can be written in the form

$$A_n e^{-\sigma_n \tau} * g(\tau)$$
$$= A_n e^{-\sigma_n \tau} \left\{ e^{-\sigma_n \tau_0} e^{\sigma_n^2/(4\alpha)} \frac{1}{\sqrt{\pi\alpha}} \left[\text{erfc}(v_1) - \text{erfc}(v_2) \right] \right\}. \tag{49}$$

Thus, to obtain the solution for the gradual phase change, one needs only multiply each of the exponentials in the residue series by the above function (which involves the pole location, $s = -\sigma_n$).

Stephan selected a five-element array and plotted the phase evolution of each oscillator as function of time when the end oscillators were injection locked to two externally derived signals 270° out of phase with each other. These signals can be simulated by choosing the Gaussian parameters to be

$$\tau_0 = 6.0 \qquad \alpha = 0.05. \tag{50}$$

This results in the phase evolution shown in Fig. 8, which is to be compared with [2, Fig. 12(a)].

Generalizing this to a 21-element array, increasing the phase difference between the injection signals to 1200°, and adjusting the Gaussian parameter α to 0.01 yields the phase distribution shown in Fig. 9. Note that, with this selection of
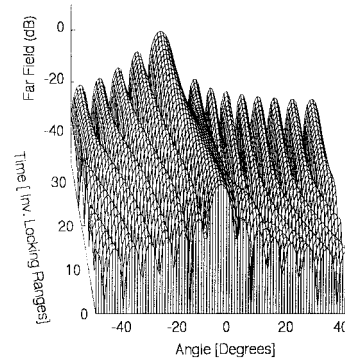

Fig. 10. Far-zone radiation pattern corresponding to the array of Fig. 9.

parameters, the phase difference between adjacent oscillators never exceeds 90°, thus, lock is maintained throughout the transient period. When the outputs of these oscillators are applied to a linear array of 21 radiating elements separated by a half-wavelength, the resulting far field is shown in Fig. 10, which illustrates the utility of the Stephan scheme in scanning the beam.

IV. CONCLUDING REMARKS

We have developed a continuum model describing the dynamics of arrays of coupled oscillators in which one or more of the oscillators is injection locked to an externally derived signal. This formalism has been used to derive several behavioral characteristics of such arrays. We have noted that, according to this theory, the frequency shift which can be induced by a step change in the frequency injected at one oscillator of the array is limited, not just to the locking range, but to the locking range divided by the number of oscillators in the array. This is consistent with the results of a full nonlinear discrete model of the array [3]. We have further shown that the response time of the array is roughly proportional to the square of the number of oscillators, which can severely limit the bandwidth of the array. This, however, can be mitigated by injection locking several of the oscillators to the same externally derived signal. In fact, as might be expected, the theory shows that if all of the oscillators are injection locked, the ensemble has the response time of a single oscillator alone. Finally, the theory was applied to the beam-steering scheme proposed and experimentally demonstrated by Stephan, in

which the end oscillators of a linear array are injection locked to signals differing in phase. Our results confirm the published results of Stephan [2].

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Ronald J. Pogorzelski (S'61–M'69–SM'82–F'95), for photograph and biography, see this issue, p. 469.

Paolo F. Maccarini, for photograph and biography, see this issue, p. 470.

Robert A. York (S'85–M'89), for photograph and biography, see this issue, p. 470.